

On the convergence of double Fourier series of functions of bounded partial generalized variation.

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ABSTRACT. The convergence of double Fourier series of functions of bounded partial Λ -variation is investigated. The sufficient and necessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the convergence of Fourier series of functions of bounded partial Λ -variation.

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On the convergence of double Fourier series

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1. CLASSES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

In 1881 Jordan [1] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ -variation, Λ -variation etc., see [2]-[5]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [6].

Let f be a real function of two variable of period 2π with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points x, y from $T := [0, 2\pi]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \quad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from T ordered in arbitrary way and let Ω be the set of all such collections E . Denote by Ω_n set of all collections of n nonoverlapping intervals $I_k \subset T$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\Lambda V_1(f) = \sup_y \sup_{E \in \Omega} \sum_n \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F \in \Omega} \sum_m \frac{|f(x, J_j)|}{\lambda_j} \quad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

Definition 1. We say that the function f has Bounded Λ -variation on $T = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function f has Bounded Partial Λ -variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \dots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \rightarrow \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$(1.1) \quad 1 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

In the case when $\lambda_n = n$, $n = 1, 2, \dots$ we say *Harmonic Variation* instead of Λ -variation and write H instead of Λ (HBV , $PHBV$, $HV(f)$, etc.).

The notion of Λ -variation was introduced by D. Waterman [4] in one dimensional case and A. Sahakian [10] in two dimensional case.

Definition 2. Let Φ -be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function f has bounded partial Φ -variation on T^2 and write $f \in PBV_\Phi$, if

$$V_\Phi^{(1)}(f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n \Phi(|f(I_i, y)|) < \infty, \quad n = 1, 2, \dots,$$

$$V_\Phi^{(2)}(f) := \sup_x \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^m \Phi(|f(x, J_j)|) < \infty, \quad m = 1, 2, \dots$$

In the case when $\Phi(u) = u^p$, $p \geq 1$, the notion of bounded partial p -variation (class PBV_p) was introduced in [8].

Theorem 1. Let $\Lambda = \{\lambda_n = n\gamma_n\}$ and $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, \dots$.

1) If

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

then $P\Lambda BV \subset HBV$.

2) If, in addition, for some $\delta > 0$

$$(1.3) \quad \gamma_n = O(\gamma_{n^{1+\delta}}) \quad \text{as } n \rightarrow \infty$$

and

$$(1.4) \quad \sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

then $P\Lambda BV \not\subset HBV$.

Proof. 1) Let $f \in P\Lambda BV$ and

$$\sum_{i,j=1}^{\infty} \frac{|f(I_i, J_j)|}{ij} = \sum_{i \leq j} \frac{|f(I_i, J_j)|}{ij} + \sum_{i > j} \frac{|f(I_i, J_j)|}{ij} := I_1 + I_2.$$

Then according to (1.2),

$$\begin{aligned} I_1 &= \sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=i}^{\infty} \frac{|f(I_i, J_j)|}{j} \\ &\leq 2 \sum_{i=1}^{\infty} \frac{1}{i} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{\lambda_j} \frac{\lambda_j}{j} \\ &\leq 2 \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{\lambda_j} \\ &\leq 2\Lambda V_2(f) \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \leq c\Lambda V_2(f) < \infty. \end{aligned}$$

Similary, $I_2 \leq c\Lambda V_1(f) < \infty$.

2) In the proof of the second statement of Theorem 1 we use the following well known lemma .

Lemma 1. *Let u_i and v_i , $i = 1, 2, \dots, j$ be two increasing (decreasing) sequences of positive numbers. Then for any rearrangement $\{\sigma(i)\}$ of the set $\{1, 2, \dots, j\}$*

$$\sum_{i=1}^j u_i v_{j-i+1} \leq \sum_{i=1}^j u_i v_{\sigma(i)} \leq \sum_{i=1}^j u_i v_i.$$

Let (1.3) and (1.4) be fulfilled and define

$$f(x, y) := \begin{cases} t_j, & x = \frac{1}{i}, y = \frac{1}{j}, j < i \leq j + m_j, i, j = 1, 2, \dots, \\ 0, & \text{otherwise} \end{cases},$$

where

$$(1.5) \quad t_j := \left(\sum_{i=1}^{m_j} \frac{1}{\lambda_j} \right)^{-1}, \quad m_j = \lceil j^{1+\delta} \rceil, \quad j = 1, 2, \dots$$

Let $x = 1/i$ and let $j(i)$ be the smallest integer satisfying

$$(1.6) \quad j(i) + m_{j(i)} \geq i.$$

Since t_j is decreasing and λ_j is increasing, using Lemma 1 we can write

$$\begin{aligned} & \sup_{F \in \Omega} \sum_{j=1}^{\infty} \frac{|f(1/i, J_j)|}{\lambda_j} \\ &= \sum_{j=j(i)}^{i-1} \frac{t_j}{\lambda_{j-j(i)}} \leq t_{j(i)} \sum_{j=1}^{i-j(i)} \frac{1}{\lambda_j} \leq t_{j(i)} \sum_{j=1}^{m_{j(i)}} \frac{1}{\lambda_j} = 1. \end{aligned}$$

Hence

$$(1.7) \quad \Lambda V_2(f) \leq 1.$$

For $y = 1/j$ we have

$$\sup_{E \in \Omega} \sum_{i=1}^{\infty} \frac{|f(I_i, 1/j)|}{\lambda_i} = t_j \sum_{i=1}^{m_j} \frac{1}{\lambda_i} = 1.$$

Consequently,

$$(1.8) \quad \Lambda V_1(f) \leq 1.$$

Combining (1.7) and (1.8) we conclude that $f \in P\Lambda BV$.

Now we prove that $f \notin H BV$. From (1.3) and (1.5) follows that

$$\sum_{i=1}^{m_j} \frac{1}{\lambda_i} = \sum_{i=1}^{m_j} \frac{1}{i^{\gamma_i}} \leq C \frac{\log m_j}{\gamma_{m_j}} \leq C \frac{\log j}{\gamma_j}.$$

Hence

$$(1.9) \quad t_j \cdot \log j \geq c \gamma_j, \quad j = 2, 3, \dots$$

and from the definition of f , (1.5) and (1.4) we obtain

$$\begin{aligned} & \sup_{E, F \in \Omega} \sum_{i, j} \frac{|f(I_i, J_j)|}{ij} \\ & \geq \sum_{j=1}^{\infty} \frac{t_j}{j} \sum_{i=j+1}^{j+m_j} \frac{1}{i} \geq c \sum_{j=1}^{\infty} \frac{t_j}{j} \log(j + m_j) \geq c \sum_{j=1}^{\infty} \frac{\gamma_j}{j} = \infty. \end{aligned}$$

Theorem 1 is proved.

Taking $\lambda_n \equiv 1$ and $\lambda_n = n$ in Theorem 1, we get

Corollary 1. $PBV \subset HBV$ and $PHBV \not\subset HBV$.

Corollary 2. Let Φ and Ψ are conjugate functions in the sense of Yung ($ab \leq \Phi(a) + \Psi(b)$) and let for some $\{\lambda_n\}$ satisfying (1)

$$(1.10) \quad \sum_{n=1}^{\infty} \Psi\left(\frac{1}{\lambda_n}\right) < \infty.$$

Then $PBV_{\Phi} \subset HBV$. In particular, $PBV_p \subset HBV$ for any $p > 1$.

Indeed, from the inequality $\frac{a}{\lambda} \leq \Phi(a) + \Psi(\frac{1}{\lambda})$ follows that $PBV_{\Phi} \subset P\Lambda BV$ under assumption (1.10), and $P\Lambda BV \subset HBV$ if (1.1) holds.

Definition 3 (see [9]). The partial modulus of variation of a function f are the functions $v_1(n, f)$ and $v_2(m, f)$ defined by

$$\begin{aligned} v_1(n, f) &:= \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n |f(I_i, y)|, \quad n = 1, 2, \dots, \\ v_2(m, f) &:= \sup_x \sup_{\{J_k\} \in \Omega_m} \sum_{k=1}^m |f(x, J_k)|, \quad m = 1, 2, \dots \end{aligned}$$

For functions of one variable the concept of the modulus variation was introduced by Chanturia [5].

Theorem 2. If $f \in B$ is bounded on T^2 and

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2,$$

then $f \in HBV$.

Proof. Using Abel transformation we can write

$$\begin{aligned}
\sum_{k=1}^m \frac{|f(x, J_k)|}{k} &= \sum_{k=1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \sum_{l=1}^k |f(x, J_l)| + \frac{1}{m} \sum_{k=1}^m |f(x, J_k)| \\
&\leq \sum_{k=1}^{m-1} \frac{1}{k^2} \left(\sum_{l=1}^k |f(x, J_l)| \right)^{1/2} \left(\sum_{l=1}^k |f(x, J_l)| \right)^{1/2} + c \\
&\leq c \sum_{k=1}^{m-1} \frac{\sqrt{k}}{k^2} \left(\sum_{l=1}^k |f(x, J_l)| \right)^{1/2} + c \\
&\leq c \sum_{k=1}^{\infty} \frac{\sqrt{v_2(k, f)}}{k^{3/2}} + c \leq c < \infty.
\end{aligned}$$

Consequently,

$$(1.11) \quad HV_2(f) < \infty.$$

Analogously, we can prove that

$$(1.12) \quad HV_1(f) < \infty.$$

Using Hardy transformation we obtain

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^m \frac{|f(I_i, J_j)|}{ij} \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \left(\frac{1}{i} - \frac{1}{i+1} \right) \left(\frac{1}{j} - \frac{1}{j+1} \right) \sum_{l=1}^i \sum_{s=1}^j |f(I_l, J_s)| \\
(1.13) \quad &+ \frac{1}{n} \sum_{j=1}^{m-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \sum_{l=1}^n \sum_{s=1}^j |f(I_l, J_s)| \\
&+ \frac{1}{m} \sum_{i=1}^{n-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \sum_{l=1}^i \sum_{s=1}^m |f(I_l, J_s)| \\
&+ \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m |f(I_i, J_j)| \\
&= I + II + III + IV.
\end{aligned}$$

Since

$$\sum_{l=1}^i \sum_{s=1}^j |f(I_l, J_s)| \leq 2i \sup_x \sum_{s=1}^j |f(x, J_s)| \leq 2iv_2(j, f)$$

and

$$\sum_{l=1}^i \sum_{s=1}^j |f(I_l, J_s)| \leq 2j \sup_y \sum_{l=1}^i |f(I_l, y)| \leq 2jv_1(i, f)$$

we can write

$$\begin{aligned}
I &\leq \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{1}{i^2 j^2} \left(\sum_{l=1}^i \sum_{s=1}^j |f(I_l, J_s)| \right)^{1/2} \left(\sum_{l=1}^i \sum_{s=1}^j |f(I_l, J_s)| \right)^{1/2} \\
(1.14) &\leq 2 \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{\sqrt{ij v_2(j, f) v_1(i, f)}}{i^2 j^2} \\
&\leq 2 \sum_{i=1}^{\infty} \frac{\sqrt{v_1(i, f)}}{i^{3/2}} \sum_{j=1}^{\infty} \frac{\sqrt{v_2(j, f)}}{j^{3/2}} < \infty,
\end{aligned}$$

$$\begin{aligned}
II &\leq \frac{1}{n} \sum_{j=1}^{m-1} \frac{1}{j^2} \left(\sum_{l=1}^n \sum_{s=1}^j |f(I_l, J_s)| \right)^{1/2} \left(\sum_{l=1}^n \sum_{s=1}^j |f(I_l, J_s)| \right)^{1/2} \\
(1.15) &\leq \frac{1}{n} \sum_{j=1}^{m-1} \frac{\sqrt{nj v_2(j, f)}}{j^2} \\
&\leq \frac{\sqrt{v_1(n, f)}}{\sqrt{n}} \sum_{j=1}^{\infty} \frac{\sqrt{v_2(j, f)}}{j^{3/2}} \leq c < \infty, n = 1, 2, \dots,
\end{aligned}$$

Analogously, we can prove that

$$(1.16) \quad III \leq c < \infty,$$

$$(1.17) \quad IV \leq 2 \sqrt{\frac{v_1(n, f)}{n} \frac{v_2(m, f)}{m}} \leq c < \infty, n, m = 1, 2, \dots$$

Combining (1.11)-(1.17) we conclude that $f \in HBV$. Theorem 2 is proved.

2. CONVERGENCE OF TWO-DIMENSIONAL TRIGONOMETRIC FOURIER SERIES

Let $f \in L^1(T^2)$, $T^2 := [0, 2\pi]^2$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{m, n=-\infty}^{+\infty} \hat{f}(m, n) e^{imx} e^{iny},$$

where

$$\hat{f}(m, n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-imx} e^{-iny} dx dy$$

are the Fourier coefficients of the function f . The rectangular partial sums are defined as follows:

$$S_{M, N}[f, (x, y)] := \sum_{m=-M}^M \sum_{n=-N}^N \hat{f}(m, n) e^{imx} e^{iny},$$

In this paper we consider convergence of **only rectangular partial sums** (convergence in the sense of Pringsheim) of double Fourier series.

We denote by $C(T^2)$ the space of continuous functions which are 2π -periodic with respect to each variable with the norm

$$\|f\|_C := \sup_{x,y \in T^2} |f(x,y)|.$$

For the function f defined on T^2 we denote by $f(x \pm 0, y \pm 0)$ the open coordinate quadrant limits (if exist) at the point (x, y) and let $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ be the arithmetic mean

$$(2.1) \quad \frac{1}{4} \{f(x+0, y+0) + f(x+0, y-0) + f(x-0, y+0) + f(x-0, y-0)\}.$$

The well known Dirichlet-Jordan theorem (see [7]) states that the Fourier series of a function $f(x)$, $x \in T$ of bounded variation converges at every point x to the value $[f(x+0) + f(x-0)]/2$. If f is in addition continuous on T the Fourier series converges uniformly on T .

Hardy [6] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function $f(x, y)$ has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges at any point (x, y) to the value $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$. If f is in addition continuous on T^2 then $S[f]$ converges uniformly on T^2 .

Theorem S (Sahakian [10]). *The Fourier series of a function $f(x, y) \in HBV$ converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ at any point (x, y) , where the quadrant limits (2.1) exist. The convergence is uniformly on any compact K , where the function f is continuous.*

Theorem S was proved in [10] under assumption that the function is continuous on some open set containing K while Sargsyan noticed in [11], that the continuity of f on the compact K is sufficient.

Analogs of Theorem S for higher dimensions can be found in [12] and [13]. Convergence of spherical and other partial sums of double Fourier series of functions of bounded Λ -variation was investigated in details by Dyachenko (see [14], [15] and references therein).

The first author [9] has proved that if f is continuous function and has bounded partial p -variation ($f \in PBV_p$) for some $p \in [1, +\infty)$ then $S[f]$ converges uniformly on T^2 . Moreover, the following is true

Theorem G (Goginava [9]). *Let $f \in C(T^2)$ and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2.$$

Then $S[f]$ converges uniformly on T^2 .

Theorems 1, 2, Corollary 2 and Theorem S imply

Theorem 3. *Let $f \in P\Lambda BV$ with*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \quad \frac{\lambda_j}{j} \downarrow 0.$$

Then $S[f]$ converges to $\sum f(x, y)$ in any point (x, y) , where the quadrant limits (2.1) exist. The convergence is uniformly on any compact K , where the function f is continuous.

Theorem 4. *Let $f \in B$ and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2.$$

Then $S[f]$ converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ in any point (x, y) , where the quadrant limits (2.1) exist. The convergence is uniformly on any compact K , where the function f is continuous.

Corollary 3. *Let $f \in B$ and $v_1(k, f) = O(k^\alpha)$, $v_2(k, f) = O(k^\beta)$, $0 < \alpha, \beta < 1$. Then $S[f]$ converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ in any point (x, y) , where the quadrant limits (2.1) exist. The convergence is uniformly on any compact K , where the function f is continuous.*

Theorem 5. *Let $f \in PBV_p$, $p \geq 1$. Then $S[f]$ converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ in any point (x, y) , where the quadrant limits in (2.1) exist. The convergence is uniformly on any compact K , where the function f is continuous.*

From Theorem 3 follows that for any $\delta > 0$ the Fourier series of the function $f \in P\left\{\frac{n}{\log^{1+\delta} n}\right\}BV$ converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ in any point (x, y) , where the quadrant limits (2.1) exist. Moreover, one can not take here $\delta = 0$ (see Theorem 6). It is interesting to compare this result with that obtained by M. Dyachenko and D. Waterman in [16].

Dyachenko and Waterman [16] introduced another class of functions of generalized bounded variation. Denoting by Γ the the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2$ we define

$$\Lambda^*V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 4 (Dyachenko, Waterman). *Let f be a real function on $T^2 := [0, 2\pi] \times [0, 2\pi]$. We say that $f \in \Lambda^*BV$ if*

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^*V(f) < \infty.$$

Theorem DW ([16]). *If $f \in \left\{\frac{n}{\log n}\right\}^*BV$, then in any point (x, y) the quadrant limits (2.1) exist and the double Fourier series of f converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$.*

Moreover, the sequence $\left\{\frac{n}{\log n}\right\}$ can not be replaced with any sequence $\left\{\frac{n\alpha_n}{\log n}\right\}$, where $\alpha_n \rightarrow \infty$.

It is easy to show (see[16]), that $\left\{\frac{n}{\log n}\right\}^* BV \subset HBV$, hence the convergence part of Theorem DW follows from Theorem S. It is essential that the condition $f \in \left\{\frac{n}{\log n}\right\}^* BV$ guaranties the existence of quadrant limits.

The next theorem in particular shows that in Theorem S the condition $HV_{1,2}(f) < \infty$ is necessary, i.e the boundedness of partial harmonic variation is not sufficient for the convergence of Fourier series of continuous function.

Theorem 6. *Let $\Lambda = \{\lambda_n = n\gamma_n\}$ where γ_n is an decreasing sequence satisfying (1.3) and (1.4). Then there exists a continuous function $f \in P\Lambda BV$ with Fourier series that diverges at $(0,0)$.*

We need the following simple lemma that easily follows from Lemma 1.

Lemma 2. *Let the function $g(t)$ be defined on T and*

$$0 = t_1 < t_2 < \dots < t_{2m} = 2\pi.$$

Suppose g is increasing on $[t_i, t_{i+1}]$ if i is odd and is decreasing, if i is even. If

$$|g(t_{i+1}) - g(t_i)| > |g(t_{i+2}) - g(t_{i+1})|, \quad i = 1, 2, \dots, 2m-2,$$

then

$$\Lambda BV(g) = \sum_{i=1}^{2m-1} \frac{|g(t_{i+1}) - g(t_i)|}{\lambda_i},$$

for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1).

Proof of Theorem 6. It is not hard to see, that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1) the class $C(T^2) \cap P\Lambda BV$ is a Banach space with the norm

$$\|f\|_{P\Lambda BV} := \|f\|_C + P\Lambda V(f).$$

Let $\Lambda = \{\lambda_n\}$ be as in Theorem 6 and denote

$$A_{i,j} = \left[\frac{\pi i}{N+1/2}, \frac{\pi(i+1)}{N+1/2} \right) \times \left[\frac{\pi j}{N+1/2}, \frac{\pi(j+1)}{N+1/2} \right).$$

Define t_j and m_j as in (1.5) and consider the function

$$(2.2) \quad f_N(x, y) = \sum_{(i,j) \in W} t_j \chi_{A_{i,j}}(x, y) \sin\left(N + \frac{1}{2}\right)x \cdot \sin\left(N + \frac{1}{2}\right)y,$$

where $\chi_A(x, y)$ is the characteristic function of the set $A \subset T^2$ and

$$W := \{(i, j) : j < i < j + m_j, \quad 1 \leq j < N_\delta\}, \quad N_\delta = \left(\frac{N}{2}\right)^{\frac{1}{1+\delta}}.$$

Each summand in the sum (2.2) is continuous on the rectangle $A_{i,j}$ and vanishes on its boundary, hence $f_N \in C(T^2)$.

Next, in view of Lemma 2, using the same arguments as in the proof of (1.7) and (1.8), we get

$$\Lambda V_1(f_N) \leq 1, \quad \Lambda V_2(f_N) \leq 1.$$

Hence $f_N \in P\Lambda BV$ and

$$(2.3) \quad \|f_N\|_{P\Lambda V} \leq 3, \quad N = 1, 2, \dots$$

Observe that $N_\delta < N$ and $j + m_j < N$, if $j < N_\delta$, hence $A_{i,j} \subset T^2$, if $(i, j) \in W$. Taking into account (1.5) and (1.9), for the square partial sum of the Fourier series of f_N at $(0, 0)$ we get

$$(2.4) \quad \begin{aligned} \pi \cdot S_{N,N}[f_N, (0, 0)] &= \int_{T^2} f_N(x, y) D_N(x) D_N(y) dx dy \\ &= \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \frac{\sin^2(N + \frac{1}{2})x \cdot \sin^2(N + \frac{1}{2})y}{4 \sin \frac{x}{2} \sin \frac{y}{2}} dx dy \\ &\geq c \sum_{j=1}^{N_\delta} \frac{t_j}{j} \sum_{i=j+1}^{j+m_j} \frac{1}{i} \geq c \sum_{j=1}^{N_\delta} \frac{t_j}{j} \log(j + m_j) \geq c \sum_{j=1}^{N_\delta} \frac{\gamma_j}{j} \rightarrow \infty. \end{aligned}$$

as $N \rightarrow \infty$, where c is an absolute constant.

Applying the Banach-Steinhaus Theorem, from (2.3) and (2.4) we obtain that there exists a continuous function $f \in P\Lambda BV$ such that

$$\sup_N |S_{N,N}[f, (0, 0)]| = \infty.$$

□

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